Western Cape Government

Education

Western Cape Education Department

Telematics
Learning Resource 2019

MATHEMATICS
Grade 12

Dear Grade 12 Learner

In 2019 there will be 8 Telematics sessions on grade 12 content and 6 Telematics sessions on grade 11 content. In grade 12 in the June, September and end of year examination the grade 11 content will be assessed. It is thus important that you compile a study timetable which will consider the revision of the grade 11 content. The program in this book reflects the dates and times for all grade 12 and grade 11 sessions. It is highly recommended that you attend both the grade 12 and 11 Telematics sessions, this will support you with the revision of grade 11 work. This workbook however will only have the material for the grade 12 Telematics sessions. The grade 11 material you will be able to download from the Telematics website. Please make sure that you bring this workbook along to each and every Telematics session.

In the grade 12 examination Trigonometry will be $\pm$ 50 marks and the Geometry $\pm$ 40 marks of the 150 marks of Paper 2.

Your teacher should indicate to you exactly which theorems you have to study for examination purposes. There are altogether 6 proofs of theorems you must know because it could be examined. These theorems are also marked with (**) in this Telematics workbook, 4 are grade 11 theorems and 2 are grade 12 theorems. At school you should receive a book called "Grade 12 Tips for Success". In it you will have a breakdown of the weighting of the various Topics in Mathematics. Ensure that you download a QR reader, this will enable you the scan the various QR codes.

At the start of each lesson, the presenters will provide you with a summary of the important concepts and together with you will work though the activities. You are encouraged to come prepared, have a pen and enough paper (ideally a hard cover exercise book) and your scientific calculator with you.

You are also encouraged to participate fully in each lesson by asking questions and working out the exercises, and where you are asked to do so, sms or e-mail your answers to the studio.

Remember:" Success is not an event, it is the result of regular and consistent hard work".

GOODLUCK, Wishing you all the success you deserve!

2019 Mathematics Telematics Program

Day	Date	Time	Grade	Subject	Topic
Term 1: 9 Jan – 15 March					
Tuesday	12 February	15:00 – 16:00	12	Mathematics	Trigonometry Revision
Wednesday	13 February	15:00 – 16:00	12	Wiskunde	Trigonometrie Hersiening
TERM 2: 2 April to 14 June					
Monday	8 April	15:00 – 16:00	12	Mathematics	Trigonometry
Tuesday	9 April	15:00 – 16:00	12	Wiskunde	Trigonometrie
Wednesday	15 May	15:00 – 16:00	11	Mathematics	Geometry
Thursday	16 May	15:00 – 16:00	11	Wiskunde	Meetkunde
Wednesday	22 May	15:00 – 16:00	12	Mathematics	Geometry
Thursday	23 May	15:00 – 16:00	12	Wiskunde	Meetkunde
Term 3: 9 July – 20 September					
Monday	29 July	15:00 – 16:00	12	Mathematics	Differential Calculus
Tuesday	30 July	15:00 – 16:00	12	Wiskunde	Differentiaalrekening
Wednesday	07 August	15:00 – 16:00	11	Mathematics	Functions
Monday	12 August	15:00 – 16:00	11	Wiskunde	Funksies
Term 4: 1 October – 4 December					
Tuesday	15 October	15:00 – 16:00	11	Mathematics	Paper 1 Revision
Wednesday	16 October	15:00 – 16:00	11	Wiskunde	Paper 2 Revision

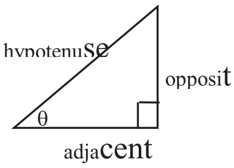
Session 1: Trigonometry

- **Definitions of trigonometric ratios:**
  - In a right-angled Δ

$\text{Sin}\,\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$

$\text{Cos}\,\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$

$\text{Tan}\,\theta = \dfrac{\text{opposite}}{\text{adjacent}}$

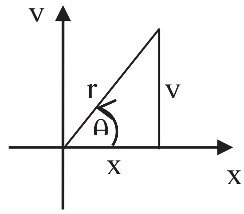


  - On a Cartesian Plane

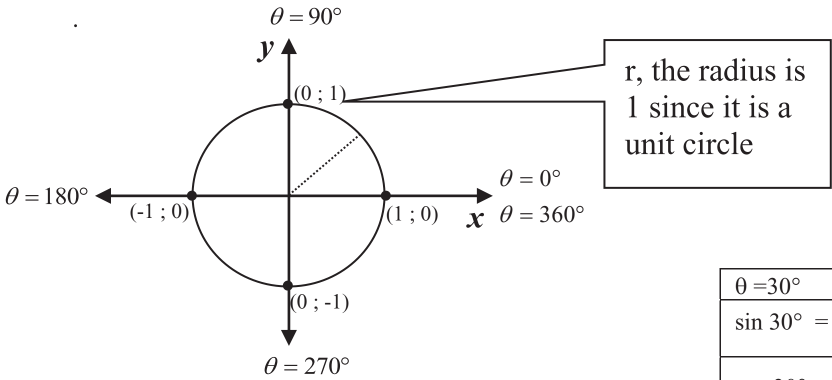
$\text{Sin}\,\theta = \dfrac{y}{r}$

$\text{Cos}\,\theta = \dfrac{x}{r}$

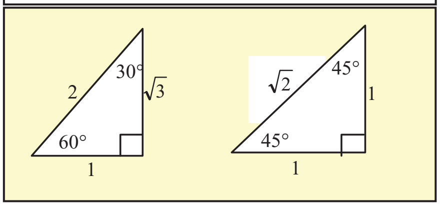
$\text{Tan}\,\theta = \dfrac{y}{x}$



- **Special Angles**
  - **0°, 90°, 180°, 270°, 360°** can be obtained from the following unit circle

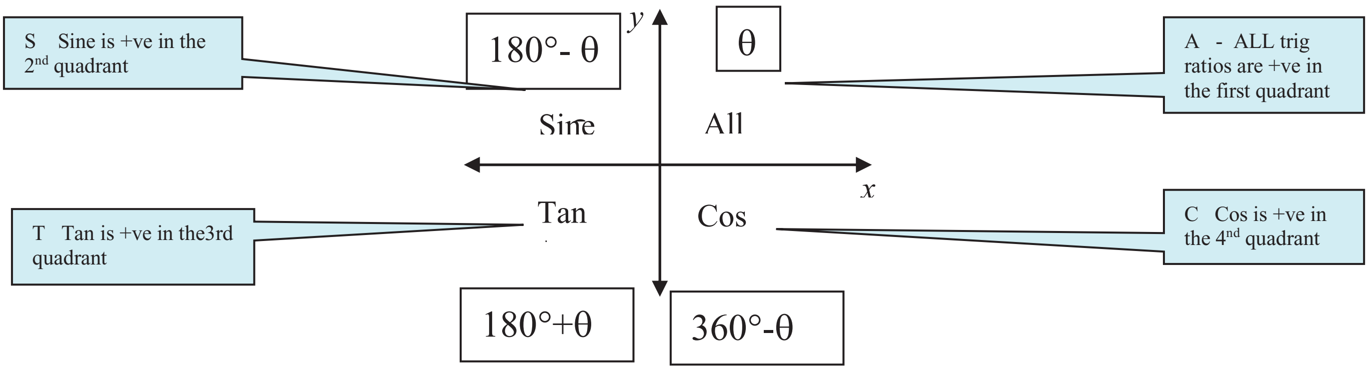


30°, 45° and 60° can be obtained from the following



θ = 30°	θ = 45°	θ = 60°
$\sin 30° = \frac{1}{2}$	$\sin 45° = \frac{1}{\sqrt{2}}$	$\sin 60° = \frac{\sqrt{3}}{2}$
$\cos 30° = \frac{\sqrt{3}}{2}$	$\cos 45° = \frac{1}{\sqrt{2}}$	$\cos 60° = \frac{1}{2}$
$\tan 30° = \frac{1}{\sqrt{3}}$	$\tan 45° = 1$	$\tan 60° = \sqrt{3}$

- The **"CAST" rule** enables you to obtain the sign of the trigonometric ratios in any of the four quadrants.



The trigonometric function of angles (180°±θ) or (360°±θ) or (-θ) → becomes → ± Trigonometric function of θ

The sign is determined by the "CAST" rule.

$(180° - \theta)$	$(180° + \theta)$	$(360° - \theta)$	$(360° + \theta)$	$(-\theta)$
$\sin(180° - \theta) = \sin\theta$	$\sin(180° + \theta) = -\sin\theta$	$\sin(360° - \theta) = -\sin\theta$	$\sin(360° + \theta) = \sin\theta$	$\sin(-\theta) = -\sin\theta$
$\cos(180° - \theta) = -\cos\theta$	$\cos(180° + \theta) = -\cos\theta$	$\cos(360° - \theta) = +\cos\theta$	$\cos(360° + \theta) = \cos\theta$	$\cos(-\theta) = +\cos\theta$
$\tan(180° - \theta) = -\tan\theta$	$\tan(180° + \theta) = +\tan\theta$	$\tan(360° - \theta) = -\tan\theta$	$\tan(360° + \theta) = \tan\theta$	$\tan(-\theta) = -\tan\theta$

- **TRIGONOMETRIC IDENTITIES**

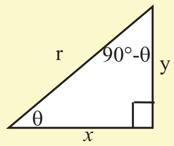
$\tan\theta = \dfrac{\sin\theta}{\cos\theta} \quad (\cos\theta \neq 0)$

$\sin^2\theta + \cos^2\theta = 1, \quad \sin^2\theta = 1 - \cos^2\theta, \quad \cos^2\theta = 1 - \sin^2\theta$

- **Co-functions or Co-ratios** →

$\sin(90° - \theta) = \cos\theta$

$\cos(90° - \theta) = \sin\theta$



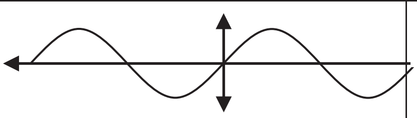
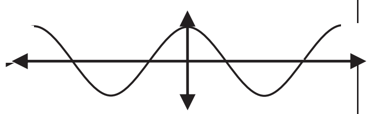
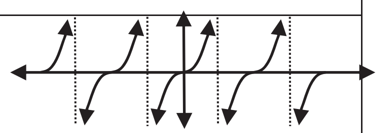
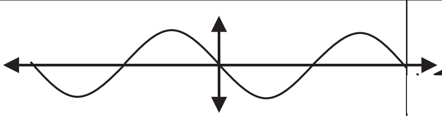
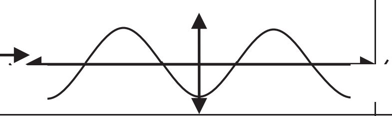
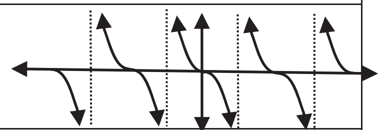
$\sin(90° + \theta) = +\cos\theta$

$\cos(90° + \theta) = -\sin\theta$

- **Trigonometric Equations**

	$\sin\theta = 0{,}707$	$\cos\theta = -0{,}866$	$\tan\theta = -1$
1. Determine the Reference angle 2. Establish in which two quadrants θ is. 3. Calculate θ in the interval $[0°;\ 360°]$ 4. Write down the general solution	Reference ∠ = $\sin^{-1}(0{,}707) = 45°$ $\therefore\ \theta = 45°$ or $\theta = 180° - 45°$ $\therefore\ \theta = 45°$ or $\theta = 135°$ $\therefore\ \theta = 45° + k360°$ or $\theta = 135° + k360°$ where $k \in Z$	Reference ∠ = $\cos^{-1}(0{,}866) = 30°$ $\therefore\ \theta = 180° - 30°$ or $\theta = 180° + 30°$ $\therefore\ \theta = 150°$ or $\theta = 210°$ $\therefore\ \theta = \pm 150°$ $\therefore\ \theta = \pm 150 + k360°$ where $k \in Z$	Reference ∠ = $\tan^{-1}(1) = 45°$ $\therefore\ \theta = 180° - 45°$ $\therefore\ \theta = 135°$ $\therefore\ \theta = 135° + k180°$ & $k \in Z$

TRIGONOMETRIC GRAPHS

	Sine Function	**Cosine Function**	**Tangent Function**
Equation	$y = a\sin k(x+p) + q$	$y = a\cos k(x+p) + q$	$y = a\tan k(x+p) + q$
Shape			
a > 0			
a < 0			
Amplitude	a	a	
Period	$\dfrac{360°}{k}$	$\dfrac{360°}{k}$	$\dfrac{180°}{k}$

SOLUTIONS OF TRIANGLES

- **Area Rule**

$$\text{Area of } \Delta ABC = \frac{1}{2}ab\sin C = \frac{1}{2}ac\sin B = \frac{1}{2}bc\sin A$$

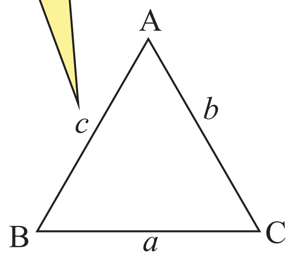
- **Sine Rule**

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \textbf{Or} \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

- **Cosine Rule**

$$a^2 = b^2 + c^2 - 2bc\cos A \quad \text{or} \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Note:
"*c*" refers to the side of the triangle opposite to angle C that is the side

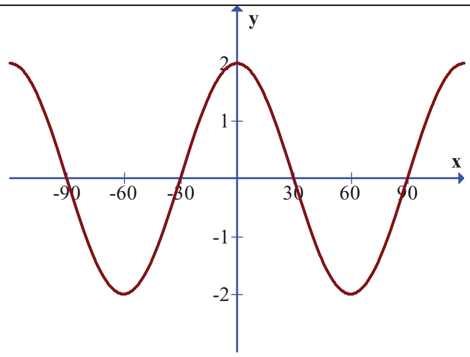
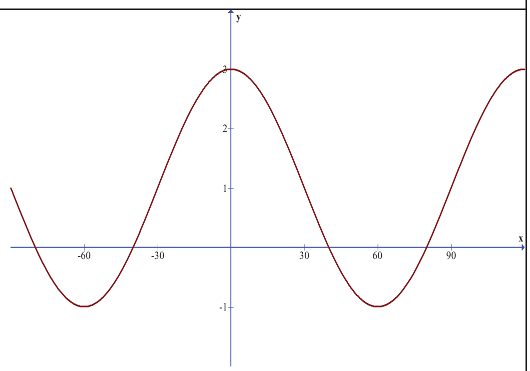


TRIGONOMETRY SUMMARY

Question type	Summary of procedure	Example question
1. Calculate the value of a trig expression without using a calculator.	Establish whether you need a rough sketch or special triangles, ASTC rules or compound angles.	1.1 If $13\cos\alpha = 5$ and $\tan\beta = -\frac{3}{4}$, $\alpha \in [0°;\ 270°]$ and $\beta \in [0°;\ 180°]$. It is given that $\sin(\alpha+\beta) = \sin\alpha.\cos\beta + \sin\beta.\cos\alpha$ Determine, without using a calculator, a) $\sin\alpha$ b) $\sin(\alpha+\beta)$. 1.2 Calculate: a) $\dfrac{\cos(-210^\circ).\sin^2 405^\circ.\cos 14^\circ}{\tan 120^\circ.\sin 104^\circ}$ b) $\sin 70°\cos 40° - \cos 70°\sin 40°$
2. If a trig ratio is given as a variable express another trig ratio in terms of the same variable.	Draw a rough sketch with given angle and label 2 of the sides. The 3rd side can then be determined using Pythagoras. Express each of the angles in question in terms of the angle in the rough sketch.	2. If $\sin 27° = q$, express each of the following in terms of *q*. a) $\sin 117°$ b) $\cos(-27°)$
3. Simplify a trigonometrical expression.	Use the ASTC rule to simplify the given expression if possible. See if any of the identities can be used to simplify it, if not see if it can be factorized. Check again if any identity can be used. This includes using the compound and double angle identities.	3. Simplify: a) $\dfrac{\cos(720° - x)\ .\ \sin(360° + x)\ .\ \tan(x - 180°)}{\sin(-x)\ .\ \cos(90° - x)}$ b) $\dfrac{\sin(90° + x)\ .\ \tan(360° + x)}{\sin(180° + x)\ .\ \cos(90° - x) + \cos(540° + x).\cos(-x)}$ c) $\dfrac{\sin^2 x\cos x + \cos^3 x}{\cos x}$ d) $\dfrac{\sin^2 x\cos x}{1 - \cos^2 x}$
4. Prove a given identity.	Simplify the one side of the equation using reduction formulae and identities until .	Prove that a) $\dfrac{\tan x\ .\ \cos^3 x}{1 - \sin^2 x + \cos^2 x} = \dfrac{1}{2}\sin x$ b) $\cos^2(180° - x) + 2\cos x\ \cos(90° + x)\tan(360° - x) = \sin^2 x + 1$
5. Solve a trig equation.	Find the reference angle by ignoring the "-"sign and finding $\sin^{-1}(0{,}435)$ Write down the two solutions in the interval $x \in [0°;\ 360°]$. Then write down the general solution of this eq. From the general solution you can determine the solution for the specified interval by using various values of *k*.	Solve for $x \in [-180°;\ 360°]$ a) $\sin x = -0{,}435$ b) $\cos 2x = 0{,}435$ c) $\tan\frac{1}{2}x - 1 = 0{,}435$

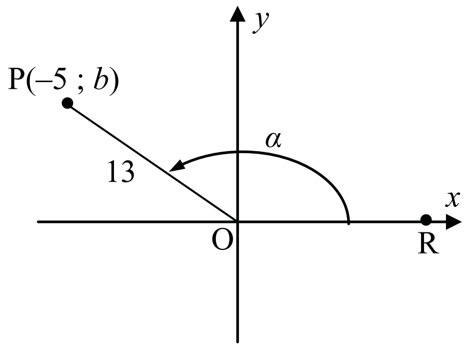
Question type	**Summary of procedure**	**Example question**
6. Sketch a trig graph.	1st sketch the trig graph without the vertical or horizontal transformation. Then shift the graph in this case 1 unit up.	Sketch b) $y = 2\cos 3x \; +1$ for $x \in [-90°;\; 120°]$ c) $y = -\sin(x+60)$ for $x \in [-240°;\; 120°]$
7.Find the area of a triangle.	If it is a right-angled triangle then $area = \frac{1}{2} base \times height$, otherwise use the area rule Area of $\Delta ABC = \frac{1}{2} ab \sin C$	ΔABC, with $\angle B = 104{,}5°$, $AB = 6cm$ and $BC = 9cm$. Calculate, correct to one decimal place area ΔABC
8. Finding an unknown side or angle in a triangle.	Draw a rough sketch with the given information. If it is not a right-angled triangle you will use either the sine or cosine rule.	a) ΔABC, with $\angle B = 104{,}5°$, $AB = 6cm$ and $BC = 9cm$. Calculate the length of AC. b) ΔABC, with $\angle C = 43{,}2°$, $AB = 4{,}5cm$ and $BC = 5{,}7cm$. Calculate the size of $\angle A$.

SKETCHING TRIG GRAPH

Calculate the period	***Write down the amplitude if it is a sine or cosine graph.***	***Identify the shape of the graph and draw a sine, cosine or tan graph with determined period and amplitude. Label the other x-intercepts. Repeat this pattern over the specified domain.***	***Now do the vertical or horizontal transformation if required.***
SKETCH $y = 2\cos 3x \; +1$ for $x \in [-90°;\; 120°]$			
Period = $\frac{360°}{3} = 120°$	Amplitude = 2		

QUESTION 1

1.1 In the figure below, the point P(–5 ; *b*) is plotted on the Cartesian plane. OP = 13 units and $R\hat{O}P = \alpha$.



Without using a calculator, determine the value of the following:

1.1.1 $\cos\alpha$ (1)

1.1.2 $\tan(180° - \alpha)$ (3)

1.2 Consider: $\dfrac{\sin(\theta - 360°)\sin(90° - \theta)\tan(-\theta)}{\cos(90° + \theta)}$

1.2.1 Simplify $\dfrac{\sin(\theta - 360°)\sin(90° - \theta)\tan(-\theta)}{\cos(90° + \theta)}$ to a single trigonometric ratio. (5)

1.2.2 Hence, or otherwise, **without using a calculator,** solve for θ if $0° \leq \theta \leq 360°$:

$$\frac{\sin(\theta - 360°)\sin(90° - \theta)\tan(-\theta)}{\cos(90° + \theta)} = 0{,}5$$ (3)

1.3 1.3.1 Prove that $\dfrac{8}{\sin^2 A} - \dfrac{4}{1 + \cos A} = \dfrac{4}{1 - \cos A}$. (5)

1.3.2 For which value(s) of A in the interval $0° \leq A \leq 360°$ is the identity in QUESTION 5.3.1 undefined? (3)

1.4 Determine the general solution of $8\cos^2 x - 2\cos x - 1 = 0$. (6)

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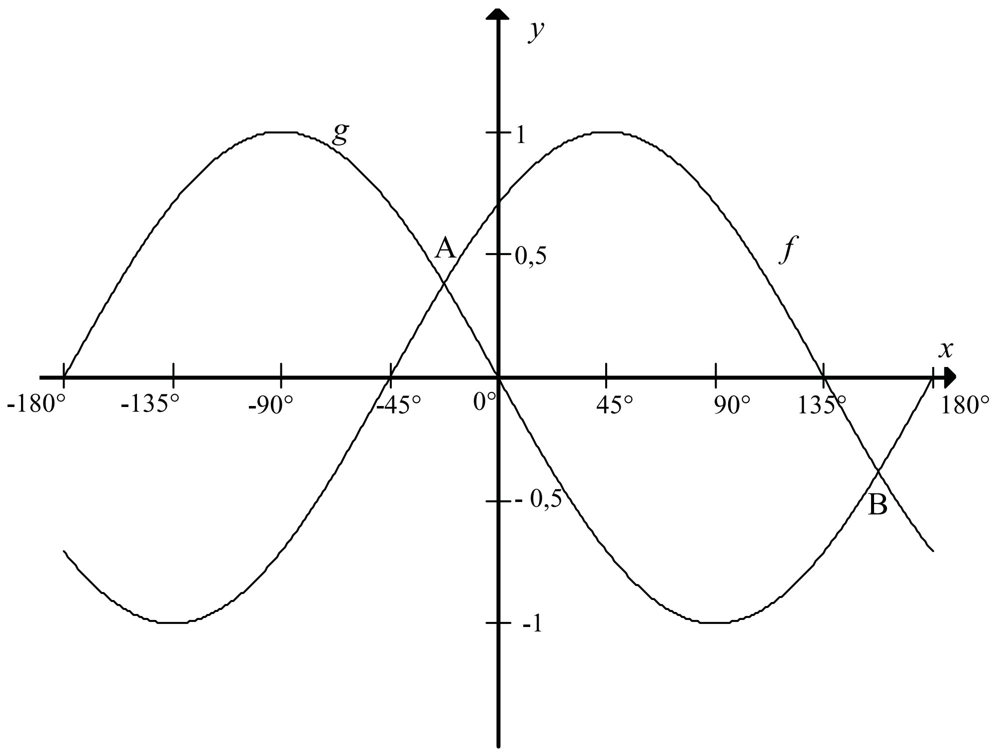
QUESTION 2

In the diagram below, the graphs of $f(x) = \cos(x + p)$ and $g(x) = q\sin x$ are shown for the interval $-180° \leq x \leq 180°$.



2.1 Determine the values of p and q. (2)

2.2 The graphs intersect at A(–22,5° ; 0,38) and B. Determine the coordinates of B. (2)

2.3 Determine the value(s) of x in the interval $-180° \leq x \leq 180°$ for which $f(x) - g(x) < 0$. (2)

2.4 The graph f is shifted 30° to the left to obtain a new graph h.

2.4.1 Write down the equation of h in its simplest form. (2)

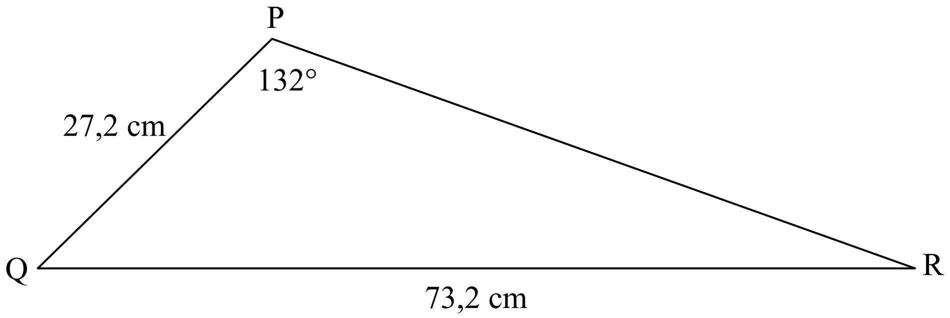
2.4.2 Write down the value of x for which h has a minimum in the interval $-180° \leq x \leq 180°$. (1)

[9]

QUESTION 3

3.1 Prove that in any acute-angled ΔABC, $\dfrac{\sin A}{a} = \dfrac{\sin C}{c}$. (5)

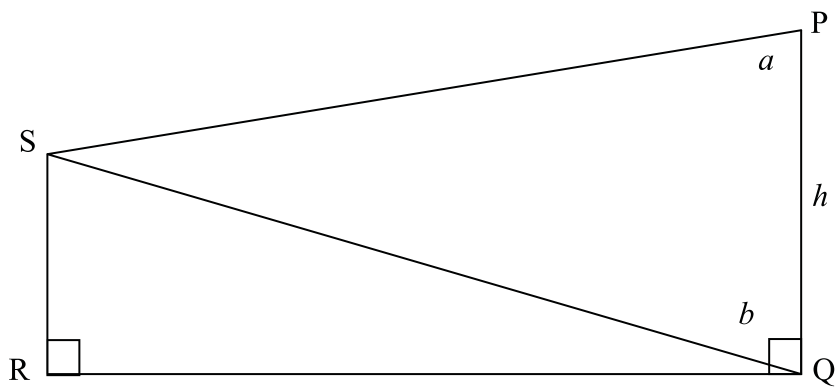
3.2 In ΔPQR, $\hat{P} = 132°$, PQ = 27,2 cm and QR = 73,2 cm.



3.2.1 Calculate the size of $\hat{R}$. (3)

3.2.2 Calculate the area of ΔPQR. (3)

3.3 In the figure below, $S\hat{P}Q = a$, $P\hat{Q}S = b$ and PQ = h. PQ and SR are perpendicular to RQ.



3.3.1 Determine the distance SQ in terms of a, b and h. (3)

3.3.2 Hence show that $RS = \dfrac{h \sin a \cos b}{\sin(a+b)}$. (3)

[17]

Session 2: TRIGONOMETRY(± 50/150 Marks)

Compound and Double Angles

In order to master this section it is best to learn the identities given below. These identities will also be given on the formulae sheet in the Examination paper.

- Compound Angle Identities:

  (a) $\cos(A-B)=\cos A\cos B+\sin A\sin B$

  $\cos(A+B)=\cos A\cos B-\sin A\sin B$

  (b) $\sin(A-B)=\sin A\cos B-\sin B\cos A$

  $\sin(A+B)=\sin A\cos B+\sin B\cos A$

When two angles are added or subtracted to form a new angle, then a compound or a double angle is formed.

- Double Angle Identities

  (c) $\sin 2A=2\sin A\cos A$

  (d) $\cos 2A=\cos^2 A-\sin^2 A$

  $=1-2\sin^2 A$

  $=2\cos^2 A-1$

Referred to as double angle formulae

What should you ensure you can do at the end of this section for examination purposes:

A. Accepting the Compound Angle formulae $\cos(A-B)=\cos A\cos B+\sin A\sin B$ use it to derive The following formulae:

$\cos(A+B)=\cos A\cos B-\sin A\sin B$

$\sin(A-B)=\sin A\cos B-\sin B\cos A$

$\sin(A+B)=\sin A\cos B+\sin B\cos A$

$\cos 2A=\cos^2 A-\sin^2 A$

$\cos 2A=1-2\sin^2 A$

$\cos 2A=2\cos^2 A-1$

$\sin 2A=2\sin A\cos A$

*You **must** remember*

Co-functions or Co-ratios

$\sin(90°-\theta)=\cos\theta$

Negative Angles

$\sin(-\theta)=-\sin\theta$

$\cos(-\theta)=+\cos\theta$

$\tan(-\theta)=-\tan\theta$

$\sin^2\theta+\cos^2\theta=1$

$\sin^2\theta=1-\cos^2\theta$

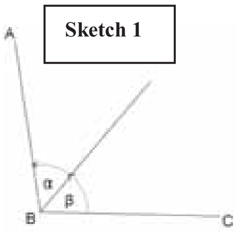
$\cos^2\theta=1-\sin^2\theta$

B. Use compound angle and double angle identities to:

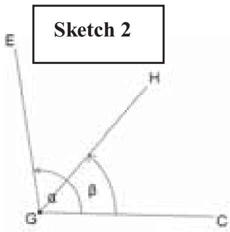
1. Evaluate an expression without using a calculator
2. Simplifying trigonometric expressions
3. Prove identities
4. Solve trigonometric equations (both specific and general solutions)

The sketches below gives a visual of compound and double angles.

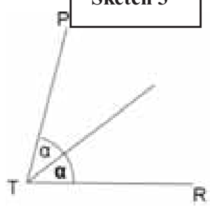
Sketch 1



Sketch 2



Sketch 3



Sketch 1: The compound angle $A\hat{B}C$ is equal to the sum of α and β. eg. $75° = 45° + 30°$

Sketch 2: The compound angle $E\hat{G}H$ is equal to the difference between α and β. eg. $15° = 60° - 45°$ or $15° = 45° - 30°$

Sketch 3: The double angle $P\hat{T}R$ is equal to the sum of α and α. eg. $45° = 22.5° + 22.5°$

Given any special angles α and β, we can find the values of the sine and cosine ratios of the angles $\alpha+\beta$, $\alpha-\beta$ and 2α.

Are you clear on the difference between a compound and double angle?

Please note:
$0°$; $30°$; $45°$; $60°$ and $90°$ are special angles, you are able to evaluate any trigonometric function of these angles without using a calculator.

Exercises: Do not use a calculator.

A. Derive each of the compound and double angle formulae in the box on the previous page.

B. 1.

1.1 Evaluate each of the following without using a calculator.

a) $\sin 75°$ b) $\cos 15°$ c) $\cos 105°$ d) $\sin 165°$

e) $\sin 36°.\cos 54° + \cos 36° \sin 54°$ f) $\cos 42°.\cos 18° - \sin 42° \sin 18°$

g) $\sin 85°.\sin 25° + \cos 85° \cos 25°$ h) $\sin 70°.\cos 40° - \cos 70° \sin 40°$

i) $2\sin 30°.\cos 30°$ j) $\dfrac{2\sin 40°.\cos 40°}{\cos 10°}$

1.2 If $\sin\alpha = \dfrac{2}{3}$, $\tan\beta = \sqrt{2}$ and α and β are acute angles determine the value of $\sin(\alpha+\beta)$.

1.3 If $\tan A = \dfrac{2}{3}$ and $90° < A < 360°$, determine without using a calculator $\cos 2A$.

2. Simplify the following expression to a single trigonometric function:

$$\frac{4\cos(-x).\cos(90°+x)}{\sin(30°-x).\cos x + \cos(30°-x).\sin x}$$

3. Prove that

a) $\cos 75° = \dfrac{\sqrt{2}(\sqrt{3}-1)}{4}$

b) $\cos(90° - 2x).\tan(180° + x) + \sin^2(360° - x) = 3\sin^2 x$

c) $(\tan x - 1)(\sin 2x - 2\cos^2 x) = 2(1 - 2\sin x \cos x)$

4. Determine the general solution for x in the following:

a) $\sin 2x.\cos 10° - \cos 2x.\sin 10° = \cos 3x$

b) $\cos^2 x = 3\sin 2x$

c) $2\sin x = \sin(x + 30°)$

Scan the QR code for revision from examination papers on this section with solutions.

ADDITIONAL QUESTIONS

1. Given $\sin\alpha = \dfrac{8}{17}$; where $90^0 \le \alpha \le 270°$

With the aid of a sketch and without the use of a calculator, calculate: (3+2+3)

a) $\tan\alpha$ b) $\sin(90°+\alpha)$ c) $\cos 2\alpha$

2. a) Using the expansions for $\sin(A+B)$ and $\cos(A+B)$, prove the identity of:

$$\frac{\sin(A+B)}{\cos(A+B)} = \frac{\tan A + \tan B}{1-\tan A.\tan B}$$ (3)

b) If $\tan(A+B) = \dfrac{\sin(A+B)}{\cos(A+B)}$, prove in any ΔABC that (4)

$\tan A.\tan B.\tan C = \tan A + \tan B + \tan C$

3. If $\sin 36°\cos 12° = p$ and $\cos 36°\sin 12° = q$, determine in terms of p and q:

a) $\sin 48°$ (3)
b) $\sin 24°$ (3)
c) $\cos 24°$ (3)

4. Show that $\sin^2 20° + \sin^2 40° + \sin^2 80° = \dfrac{3}{2}$

(**HINT:** $40° = 60° - 20°$ **and** $80° = 60° + 20°$ (7)

5. Given: $f(x) = 1 + \sin x$ and $g(x) = \cos 2x$

Calculate the points of intersection of the graphs f and g for $x \in [180° ; 360°]$ (7)

6. Given that $\sin\theta = \dfrac{1}{3}$, calculate the numerical value of $\sin 3\theta$, **WITHOUT** using a calculator. (5)

7. Prove that, for any angle A:

$$\frac{4\sin A\cos A\cos 2A\sin 15°}{\sin 2A(\tan 225° - 2\sin^2 A)} = \frac{\sqrt{6}-\sqrt{2}}{2}$$ (6)

8. Solve for x if $2\cos x = \tan 2x$ and $x \in [-90° ; 90°]$. Show ALL working details. (8)

9. If $\cos\beta = \dfrac{p}{\sqrt{5}}$; where $p < 0$ and $\beta \in [0^0;\ 90^0]$, determine, using a diagram, an expression in terms of p for:

a) $\tan\beta$ (4)
b) $\cos 2\beta$ (3)

10.1 If sin 28° = a and cos 32°= b, determine the following in terms of a and/or b:

a) $\cos 28°$ b) $\cos 64°$ c) $\sin 4°$ (2+3 +4)

10.2 Prove without the use of a calculator, that if sin 28° = a and cos 32° = b, then (4)

$b\sqrt{1-a^2} - a\sqrt{1-b^2} = \dfrac{1}{2}$.

Revision: Grade 11 Geometry Theorems and Converses

The proofs of the theorems marked with (**) must be studied because it could be examined. The part in bold in bracket is the abbreviation for the theorem, which we use as reasons when writing up geometry solutions.

1	**Theorem****	The line drawn from the centre of a circle perpendicular to a chord bisects the chord; **(line from centre ⊥ to chord)**
	Converse	The line from the centre of a circle to the midpoint of a chord is perpendicular to the chord. **(line from centre to midpt of chord)**
		The perpendicular bisector of a chord passes through the centre of the circle; **(perp bisector of chord)**
2	**Theorem****	The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle (on the same side of the chord as the centre); **(∠ at centre = 2 ×∠ at circumference)**
	Corollary	1. Angle in a semi-circle is 90^0 **(∠s in semi circle)** 2. Angles subtended by a chord of the circle, on the same side of the chord, are equal **(∠s in the same seg)** 3. Equal chords subtend equal angles at the circumference **(equal chords; equal ∠s)** 4. Equal chords subtend equal angles at the centre **(equal chords; equal ∠s)** 5. Equal chords in equal circles subtend equal angles at the circumference of the circles. **(equal circles; equal chords; equal ∠s)**
	Corollary **Converse**	1. If the angle subtended by a chord at the circumference of the circle is 90°, then the chord is a diameter. **(converse ∠s in semi circle)** 2. If a line segment joining two points subtends equal angles at two points on the same side of the line segment, then the four points are concyclic.
3	**Theorem****	The opposite angles of a cyclic quadrilateral are supplementary; **(opp ∠s of cyclic quad)**
	Converse	If the opposite angles of a quadrilateral are supplementary then the quadrilateral is a cyclic quadrilateral. **(opp ∠s quad sup OR converse opp ∠s of cyclic quad)**
	Corollary	The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle of the quadrilateral. **(ext ∠ of cyclic quad)**
	Corollary **Converse**	If the exterior angle of a quadrilateral is equal to the interior opposite angle of the quadrilateral, then the quadrilateral is cyclic. **(ext ∠ = int opp ∠ OR converse ext ∠ of cyclic quad)**
4	**Theorem**	The tangent to a circle is perpendicular to the radius/diameter of the circle at the point of contact. **(tan ⊥ radius)**
	Converse	If a line is drawn perpendicular to a radius/diameter at the point where the radius/diameter meets the circle, then the line is a tangent to the circle. **(line ⊥ radius)**
5	**Theorem**	Two tangents drawn to a circle from the same point outside the circle are equal in length. **(Tans from common pt OR Tans from same pt)**
6	**Theorem****	The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment. **(tan chord theorem)**
	Converse	If a line is drawn through the end-point of a chord, making with the chord an angle equal to an angle in the alternate segment, then the line is a tangent to the circle. **(converse tan chord theorem OR ∠ between line and chord)**

Scan the QR code for grade 11 geometry revision with solutions.

Session 3 Grade 12 Geometry

The Grade 11 geometry entails the circle geometry theorems dealing with angles in a circle, cyclic quadrilaterals and tangents. The Grade 12 geometry is based on ratio and proportion as well as similar triangles. Grade 11 geometry is especially important in order to do the grade 12 Geometry hence this work must be thoroughly understood and regularly practiced to acquire the necessary skills. The grade 11 geometry is summarized on the previous page.
Below are **Grade 12 Theorems**, Converse Theorems and their Corollaries which you must know. The proofs of the theorems marked with (**) must be studied because it could be examined.

1	**Theorem****	A line drawn parallel to one side of a triangle divides the other two sides proportionally. **(line || one side of Δ OR prop theorem; name || lines)**
	Converse	If a line divides two sides of a triangle in the same proportion, then the line is parallel to the third side. **(line divides two sides of Δ in prop)**
	Theorem**	If two triangles are equiangular, then the corresponding sides are in proportion (and consequently the triangles are similar) **(||| Δs OR equiangular Δs)**
	Converse	If the corresponding sides of two triangles are proportional, then the triangles are equiangular (and consequently the triangles are similar). **(Sides of Δ in prop)**

PROPORTIONALITY

Two variables are **proportional** if there is a constant **ratio** between them.

Ratio A ratio describes the relationship between two quantities which have the same units. We can use ratios to compare lengths, age, etc. A ratio is a comparison between two quantities of the same kind and has no units.

Example 1: if the length of the base of a triangle is 200 cm and the height is 40 cm, then we can express the ratio between the length of the base and the height of the triangle:

$Length\ of\ base : height$

$200 : 40$

$5 : 1$

$$\frac{lenght\ of\ base}{height} = \frac{200}{40} = \frac{5}{1}$$

A ratio written as a fraction is usually given in its simplest form.

Example: If $\frac{AB}{CD} = \frac{5}{10} = \frac{1}{2}$

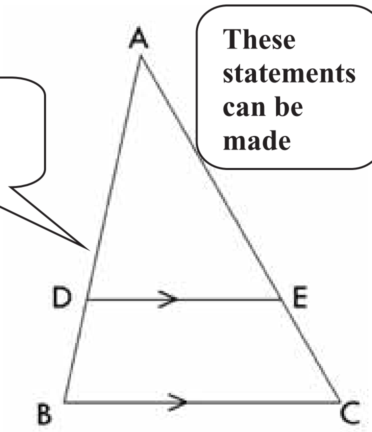
And $\frac{KL}{MN} = \frac{4}{8} = \frac{1}{2}$

$\therefore \frac{AB}{CD} = \frac{KL}{MN}$

If two or more **ratios** are equal to each other, then we say that they are in the same **proportion**.

Triangle **Proportionality Theorem**.
If a line parallel to one side of a triangle intersects the other two sides of the triangle, then the line divides these two sides **proportionally**.

Given:



These statements can be made

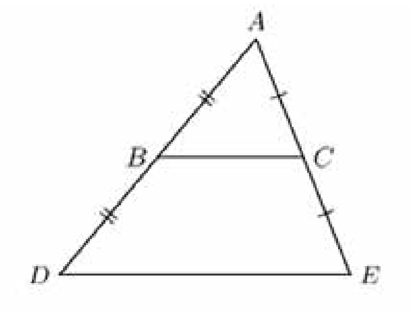
Statement	Reason
$\frac{AD}{DB}=\frac{AE}{EC}$	prop theorem DE ‖ BC
$\frac{AB}{DB}=\frac{AC}{EC}$	prop theorem DE ‖ BC
$\frac{AD}{AB}=\frac{AE}{AC}$	prop theorem DE ‖ BC

The theorem is the reason,

The proportionality theorem written as a reason in short.

SPECIAL CASE OF THE CONVERSE PROPORTIONALITY THEOREM: THE MID-POINT THEOREM

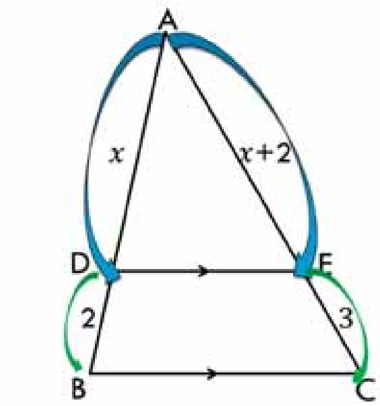
A corollary of the proportion theorem is the mid-point theorem: the line joining the mid-points of two sides of a triangle is parallel to the third side and equal to half the length of the third side.



If AB = BD and AC = CE, then BC‖DE and BC = ½DE.

We also know that $\frac{AC}{CE}=\frac{AB}{BD}$

APPLYING THE PROPRTIONALITY THEOREM:
EXAMPLE 1
In the diagram below, ΔABC has D on AB and E on AC such that DE ‖ BC.
DB = 2 units, EC = 3 units, AD = x units and AE = $x+2$ units.
Determine the value of x.

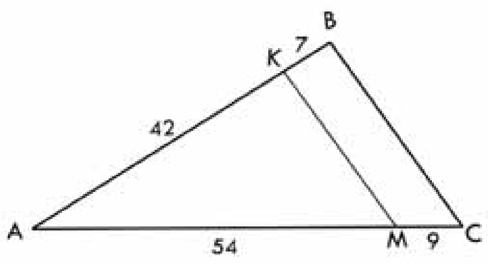


Statement	Reason
$\frac{AD}{DB}=\frac{AE}{EC}$	prop theorem DE ‖ BC
$\frac{x}{2}=\frac{x+2}{3}$	
$2(x+2)=3x$ $2x+4=3x$ $4=x$	

CONVERSE OF THE PROPORTIONALITY THEOREM:

EXAMPLE 2

In the diagram : KB = 7 units; AK = 42 units; AM = 54units and MC = 9 units.
Prove that KM is parallel to BC.



We need to prove that KM divide the sides of the Δ ABC proportionally (*in other words*: $\frac{AK}{KB} = \frac{AM}{MC}$) :

Let's investigate:

$$\frac{AK}{KB} = \frac{42}{7} = 6$$

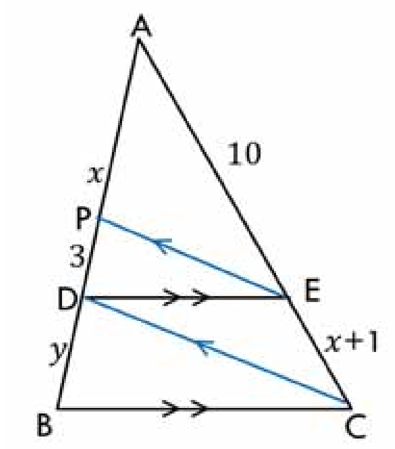
$$\frac{AM}{MC} = \frac{54}{9} = 6$$

$\therefore$ KM || BC

This could come in handy if you want to prove TWO lines are parallel!

EXAMPLE 3

In the diagram, ΔABC has D and P on AB and E on AC such that DE || BC and PE || DC
DB = y units, DP = 3 units, AP = x units, AE = 10 units and AE = $x + 1$ units.
Determine the value of x.



$\frac{AP}{DP} = \frac{AE}{EC}$ prop theorem PE||DC

$$\frac{x}{3} = \frac{10}{x+1}$$

$$x(x+1) = 30$$

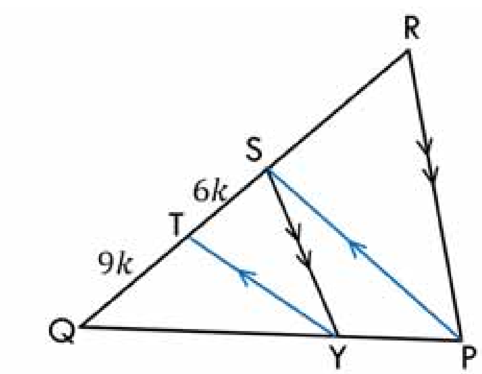
$$x^2 + x - 30 = 0$$

$$(x+6)(x-5) = 0$$

$$x \neq -6 \ or \ x = 5$$

EXAMPLE 4

In the diagram below, ΔPQR has T and S on RQ and Y on QP such that TY|| SP and SY|| PR
If $\frac{QT}{TS} = \frac{9}{6}$; determine the ratio of $\frac{TS}{SR}$

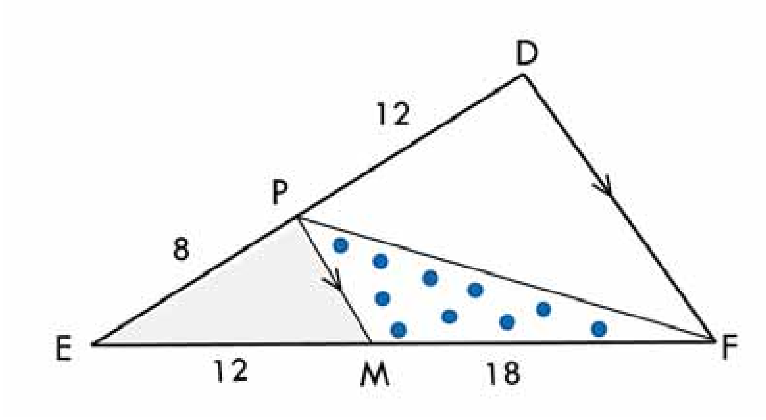


Statement	**Reason**
$\frac{QY}{YP} = \frac{QT}{TS}$	prop theorem TY|| SP
$\frac{QX}{XP} = \frac{9k}{6k} = \frac{3}{2}$	
$\frac{QY}{YP} = \frac{QS}{SR}$	prop theorem SY|| PR
$\frac{3}{2} = \frac{9k+6k}{SR}$	
$3SR = 30k$	
$SR = 10k$	
$\frac{TS}{SR} = \frac{6k}{10k} = \frac{6}{10}$	

AREA OF TRIANGLES IN PROPORTIONALITY PROBLEMS:

EXAMPLE 5

In the diagram is ΔEFD with PM parallel to DF.
PD=12 units, EP = 8 units, EM =12 units and MF=18units



5.1 Determine the ratio of: $\frac{area\,\Delta PEM}{area\,\Delta PMF}$

5.2 Determine the ratio of: $\frac{area\,\Delta PEM}{area\,\Delta DEF}$

- There are TWO known formulas for the area of a Δ.
- We have to decide which formula works best in a given question.

1) Area of $\Delta = \frac{1}{2} \times base \times height$ → *use when two Δs have a common height.*

2) Area of $\Delta = \frac{1}{2} \times ab\ sinC$ → *use when two Δs have a common angle.*

5.1

$$\frac{area\,\Delta PEM}{area\,\Delta PMF} = \frac{\frac{1}{2} \times EM \times h_P}{\frac{1}{2} \times MF \times h_P} \ldots .common\ height$$

$$\frac{area\,\Delta PEM}{area\,\Delta PMF} = \frac{EM}{MF}$$

$$\frac{area\,\Delta PEM}{area\,\Delta PMF} = \frac{12}{18}$$

$$\frac{area\,\Delta PEM}{area\,\Delta PMF} = \frac{2}{3}$$

Always simplify

5.2

$$\frac{area\,\Delta PEM}{area\,\Delta DEF} = \frac{\frac{1}{2} \times EM \times PE \times sinE}{\frac{1}{2} \times EF \times ED \times sinE} \ldots common\ angle$$

$$\frac{area\,\Delta PEM}{area\,\Delta DEF} = \frac{EM \times PE}{EF \times ED}$$

$$\frac{area\,\Delta PEM}{area\,\Delta DEF} = \frac{12 \times 8}{30 \times 20}$$

$$\frac{area\,\Delta PEM}{area\,\Delta DEF} = \frac{96}{600}$$

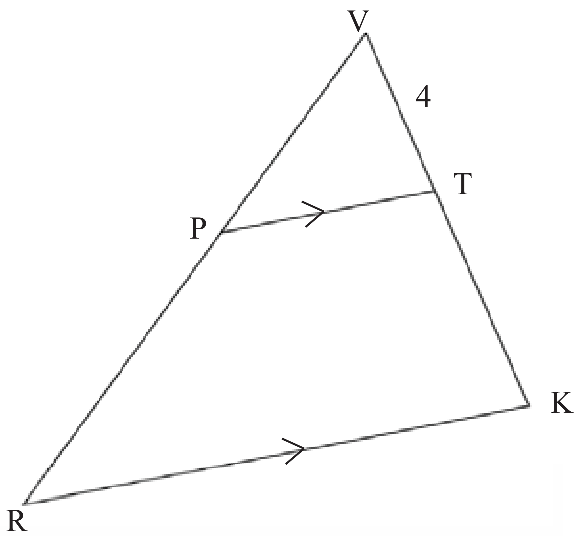
$$\frac{area\,\Delta PEM}{area\,\Delta DEF} = \frac{4}{25}$$

Scan the QR code for revision from examination papers on the grade 12 geometry theorems with solutions.

EXERCISE 1

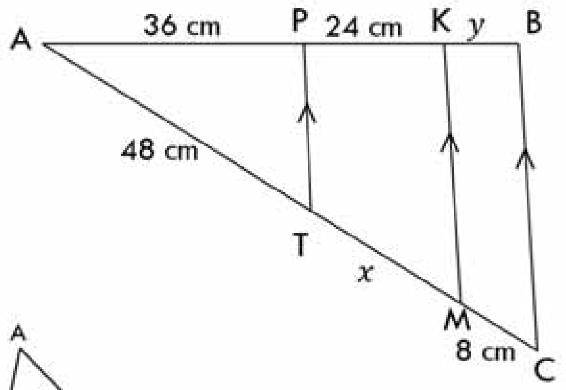
QUESTION 1

In the diagram below, ΔVRK has P on VR and T on VK such that PT ∥ RK. VT = 4 units, PR = 9 units, TK = 6 units and VP = $2x - 10$ units.
Calculate the value of x.



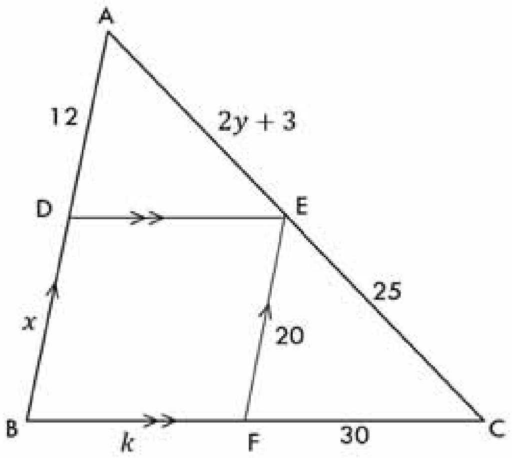
QUESTION 2

In the diagram, ΔABC has P and K on AB and T and M on AC such that PT ∥ KM ∥ BC.
AP = 36cm, PK = 24cm, AT = 48cm; MC = 8cm, KB= y and TM = x
Calculate the value of x and y.



QUESTION 3

In the diagram below, ΔABC has D on AB ; F on BC and E on AC such that DE ∥ BC and EF∥ AB .
AD = 12 units, EC = 25 units EF = 20 units and FC = 30 units.
DB = x ; BF = k and AE = $2y + 3$ $units$.
Calculate the value of x, y and k.



QUESTION 4

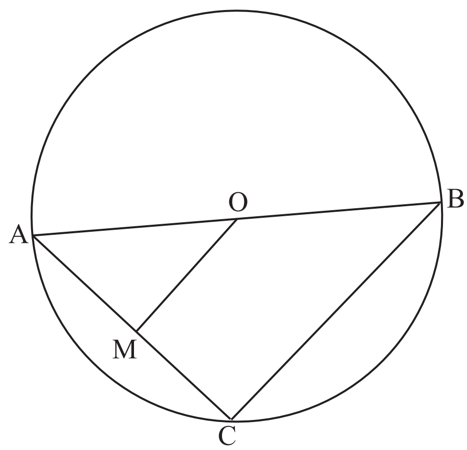
O is the centre of the circle below. OM ⊥ AC. The radius of the circle is equal to 5 cm and BC = 8 cm.

4.1 Write down the size of $B\hat{C}A$.

4.2 Calculate:

4.2.1 The length of AM, with reasons.

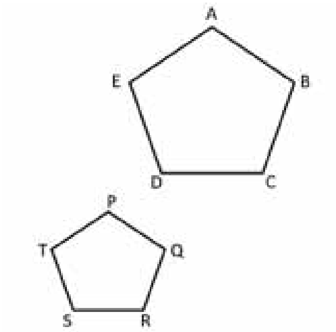
4.2.2 Area ΔAOM : Area ΔABC



SIMILARITY

Two polygons with the same number of sides are similar if:
1) All pairs of corresponding angles are equal **and**
2) All pairs of corresponding sides are in the same ratio.

The symbol for similarity is: |||

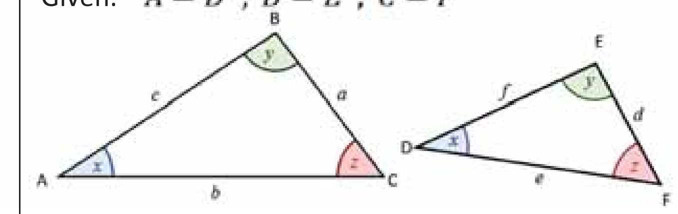


ABCDE is similar to PQRST if:

1) $\hat{A} = \hat{P}$; $\hat{B} = \hat{Q}$ $\hat{C} = \hat{R}$ $\hat{D} = \hat{S}$ $\hat{E} = \hat{T}$ and
2) $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CD}{RS} = \frac{DE}{ST} = \frac{EA}{TP}$

✓ Both conditions must be true for two polygons to be similar.

Theorem**

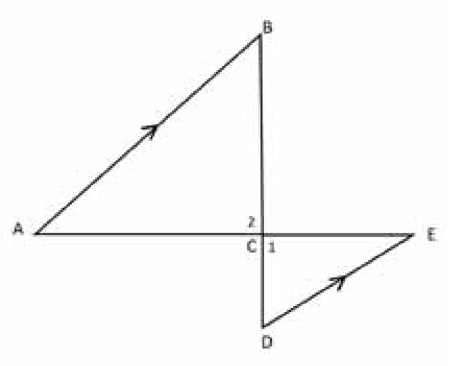
If two triangles are equiangular, then the corresponding sides are in proportion (and consequently the triangles are similar) **(||| Δs OR equiangular Δs)**	
Given: $\hat{A} = \hat{D}$; $\hat{B} = \hat{E}$; $\hat{C} = \hat{F}$ 	Then $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$
Converse	
If the corresponding sides of two triangles are proportional, then the triangles are equiangular (and consequently the triangles are similar). **(Sides of Δ in prop)**	
Given: **ΔABC** and **ΔDEF** with $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$	Then $\hat{A} = \hat{D}$; $\hat{B} = \hat{E}$; $\hat{C} = \hat{F}$
Note:	

Be careful to correctly label similar triangles. The angles that are equal must be in the same position:

ΔABC ||| ΔDEF (∠, ∠, ∠) → $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$

EXAMPLE 1

In the diagram is $AB \parallel DE$. Prove that $\Delta ABC \parallel\!| \Delta EDC$



In ΔABC and ΔDEC:

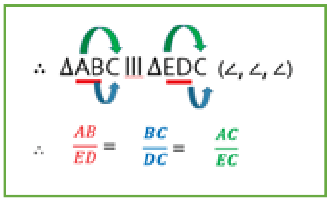
1) $\hat{A} = \hat{E}$ (alt $\angle s$; $AB \parallel DE$)

2) $\hat{B} = \hat{D}$ (alt $\angle s$; $AB \parallel DE$)

$\therefore \Delta ABC \; ||| \; \Delta EDC$ ($\angle, \angle, \angle$)

• Given that figures are similar; we can deduce information about their corresponding parts that we didn't previously know.

$$\therefore \frac{AB}{ED} = \frac{BC}{DC} = \frac{AC}{EC}$$

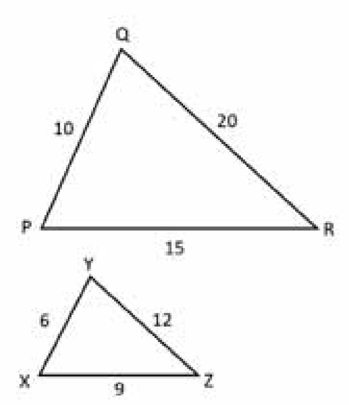


EXAMPLE 2

In the diagram is

$PR = 15$; $QR = 20$; $XZ = 9$; $YZ = 12$ and $XY = 6$ units.

Prove that $\Delta XYZ \parallel\!| \Delta PQR$ and that $\hat{Q} = \hat{Y}$.



In ΔPQR and ΔXYZ :

1) $\frac{QP}{YX} = \frac{10}{6} = \frac{5}{3}$

2) $\frac{QR}{YZ} = \frac{20}{12} = \frac{5}{3}$

3) $\frac{PR}{XZ} = \frac{15}{9} = \frac{5}{3}$

$$\therefore \frac{QP}{YX} = \frac{QR}{YZ} = \frac{PR}{XZ}$$

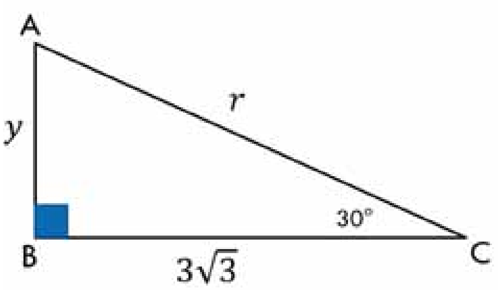
$\therefore \Delta XYZ \; ||| \; \Delta PQR$ (sides in the same proportion)

Given that figures are similar; we can deduce information about their corresponding angless that we didn't previously know.

$\therefore \hat{X} = \hat{P}$ and $\hat{Y} = \hat{Q}$ and $\hat{Z} = \hat{R}$

EXAMPLE 3 :

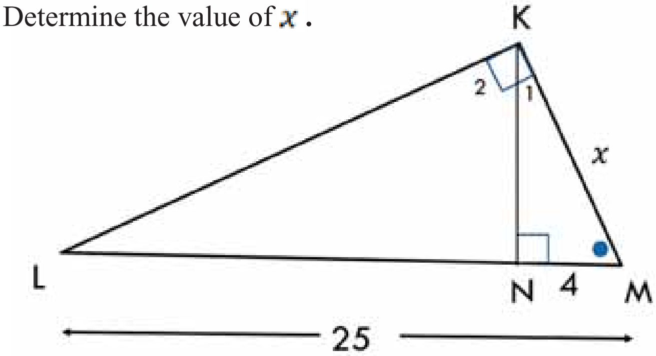
Given the sketch below determine the values of y and r without using a calculator.



EXAMPLE 4:

In the diagram is LK $\perp$ KM and KN $\perp$ LM

Determine the value of x .

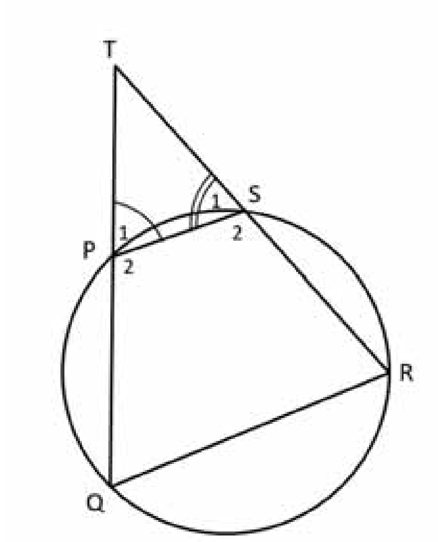


EXAMPLES WITH CIRCLE GEOMETRY.

EXERCISE 1

QUESTION 1

In the diagram below P, S, R and Q are points on the circumference of the circle. QP produced meets RS produced at T.



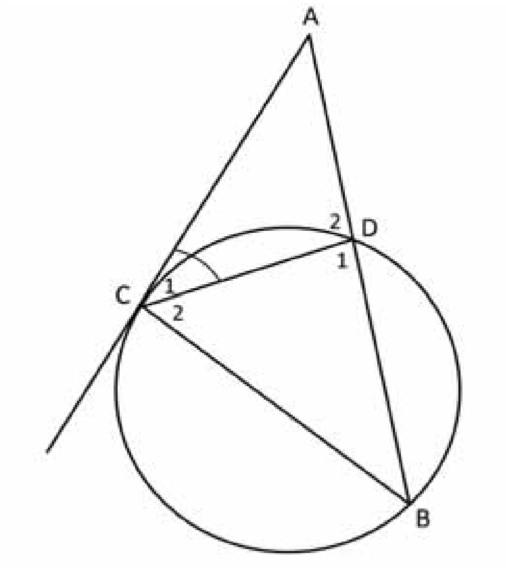
1.1 Prove that $\Delta TPS ||| \Delta TRQ$

1.2 Show that TP. TQ = TS. TR

1.3 Hence, or otherwise, prove that $PQ = \frac{TR.TS - TP^2}{TP}$

QUESTION 2

In the diagram below, CA is a tangent to the circle.



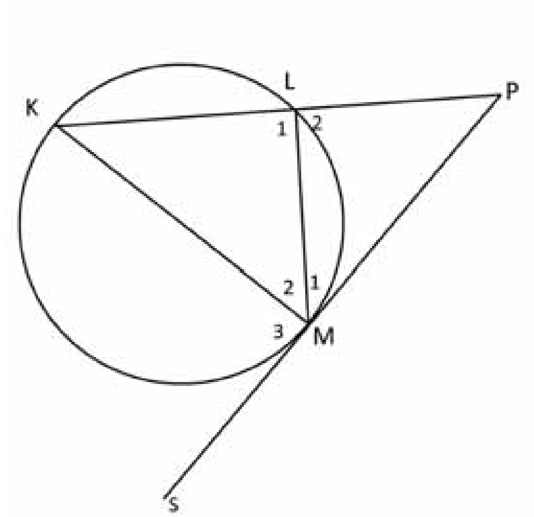
Prove, with reasons, that:

2.1 $\Delta ACD ||| \Delta ABC$

2.2 CD.AC=BC.AD

QUESTION 3

In the diagram below SP is a tangent to the circle. KM is the diameter of the circle.



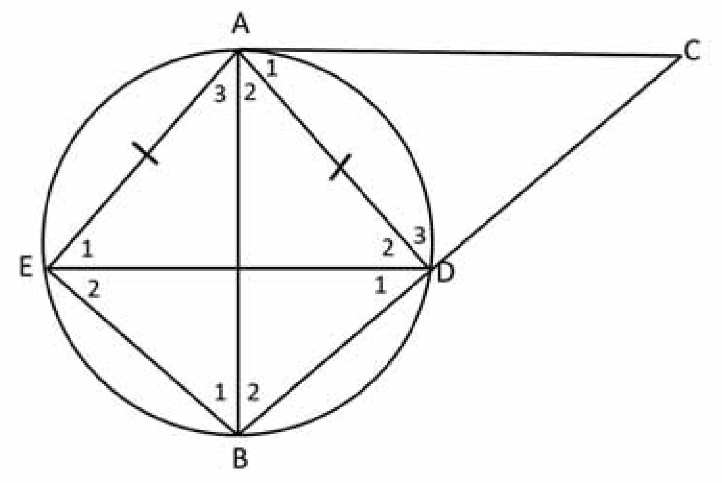
Prove, with reasons, that:

3.1 $\Delta KLM ||| \Delta MLP$

3.2 $ML^2 = KL.\ LP$

QUESTION 4

In the diagram below CA is a tangent to the circle with EA = AD.



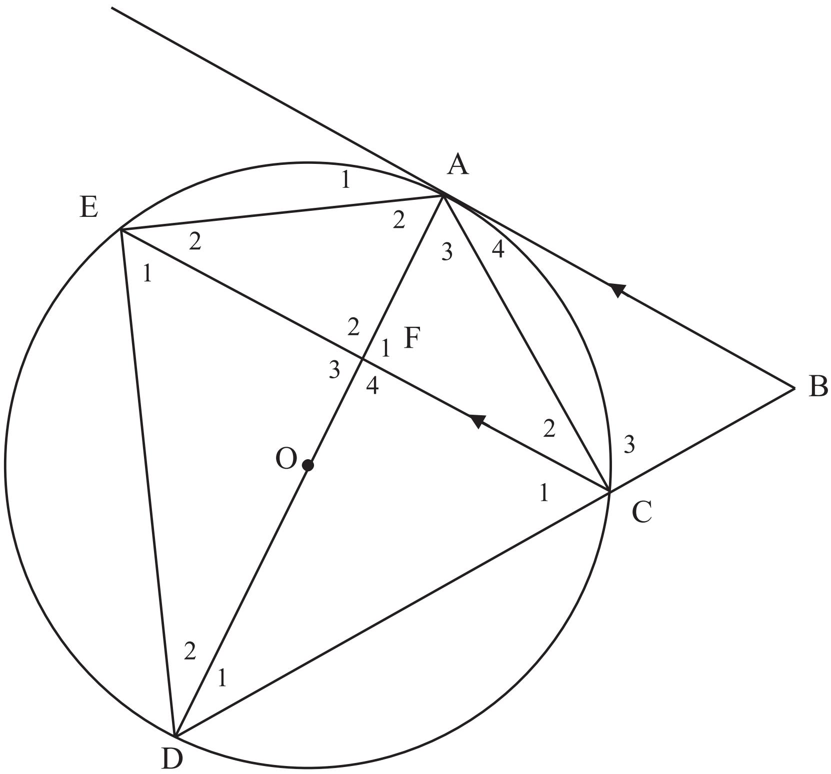
Prove, with reasons, that:

4.1 $\Delta ADC \mid\mid\mid \Delta BEA$

4.2 $AD^2 = BE.DC$

QUESTION 5

In the figure below, AB is a tangent to the circle with centre O. AC = AO and BA || CE.
DC produced, cuts tangent BA at B.



Prove, with reasons, that:

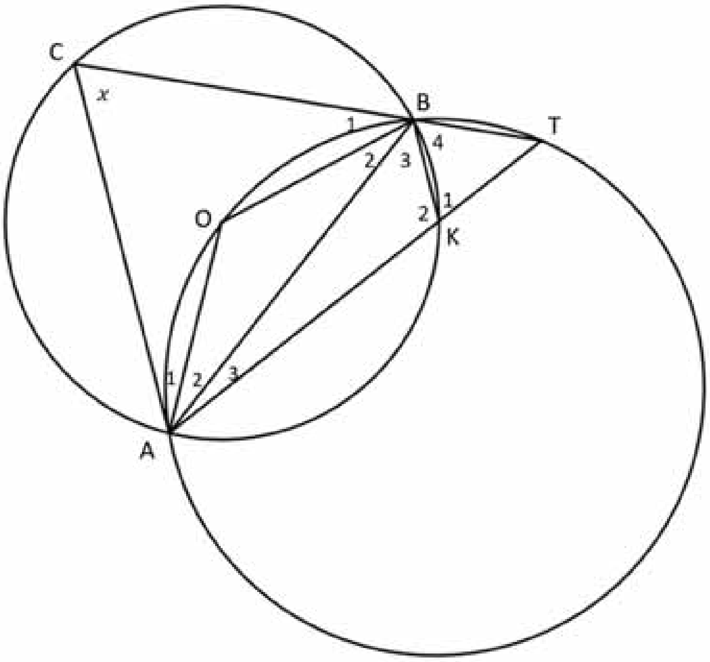
5.1 $\hat{C}_2 = \hat{D}_1$

5.2 $\Delta ACF \mid\mid\mid \Delta ADC$

5.3 $AD = 4AF$

QUESTION 6

In the figure below, O is the centre of the circle CAKB. AK produced intersects circle AOBT at T. $A\hat{C}B = x$



6.1 Prove, with reasons, that:

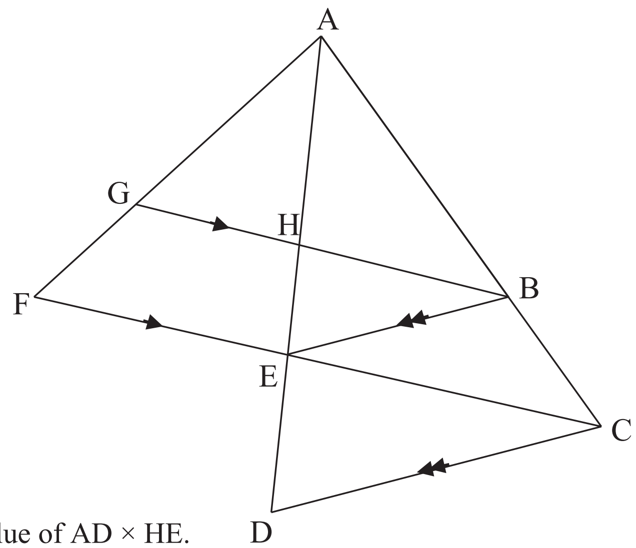
6.1.1 $\hat{T} = 180° - 2x$

6.1.2 AC || KB.

6.1.3 ΔBKT ||| ΔCAT

6.2 If AK : KT = 5 : 2, determine the value of $\frac{AC}{KB}$

7. In the figure below, GB || FC and BE || CD. AC = 6 cm and $\frac{AB}{BC} = 2$.



7.1 Calculate with reasons:

a) AH : ED

b) $\frac{BE}{CD}$

7.2 If HE = 2 cm, calculate the value of AD × HE.

Session 4: Grade 12 Calculus

Cubic Graphs. In this lesson you will work through 3 types of questions regarding graphs

1. Drawing cubic graphs

2. Given the graphs, answer interpretive questions

3. Given the graphs of the derivative, answer interpretive questions

1.1 Given: $f(x) = 2x^3 - x^2 - 4x + 3$

1.1.1 Show that $(x-1)$ is a factor of $f(x)$. (2)

1.1.2 Hence factorise $f(x)$ completely. (2)

1.1.3 Determine the co-ordinates of the turning points of f. (4)

1.1.4 Draw a neat sketch graph of f indicating the co-ordinates of the turning points as well as the x-intercepts. (4)

1.1.5 For which value of x will f have a point of inflection? (4)

[16]

1.2 Given $f(x) = x^3 + x^2 - 5x + 3$

1.2.1 Draw a sketch graph of $f(x)$. (7)

1.2.2 For which value(s) of x is $f(x)$ increasing? (2)

1.2.3 Describe one transformation of $f(x)$ that, when applied, will result in $f(x)$ having two unequal positive real roots. (2)

1.2.4 Give the equation of g if g is the reflection of f in the y-axis. (3)

1.2.5 Determine the average rate of change of f between the points (0 ; 3) and (1;0). (2)

1.2.6 Determine the equation of the tangent to the f when x = -2. (4)

1.2.7 Prove that the tangent in 1.2.6 will intersect or touch the curve of f at two places. (4)

1.3 A cubic function f has the following properties:

- $f\left(\frac{1}{2}\right) = f(3) = f(-1) = 0$
- $f'(2) = f'\left(-\frac{1}{3}\right) = 0$

f decreases for $x \in \left[-\frac{1}{3}\,;\,2\right]$ only

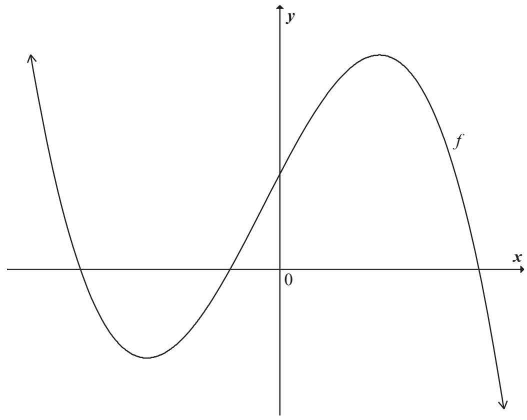
Draw a possible sketch graph of f, clearly indicating the x-coordinates of the turning points and ALL the x-intercepts. (4)

1.4 If f is a cubic function with:

- $f(3) = f'(3)$,
- $f(0) = 27$,
- $f''(x) > 0$ when $x < 3$ and $f''(x) < 0$ when $x > 3$ (3)

draw a sketch graph of f indicating ALL relevant points.

2.1 The graph of the function $f(x) = -x^3 - x^2 + 16x + 16$ is sketched below.

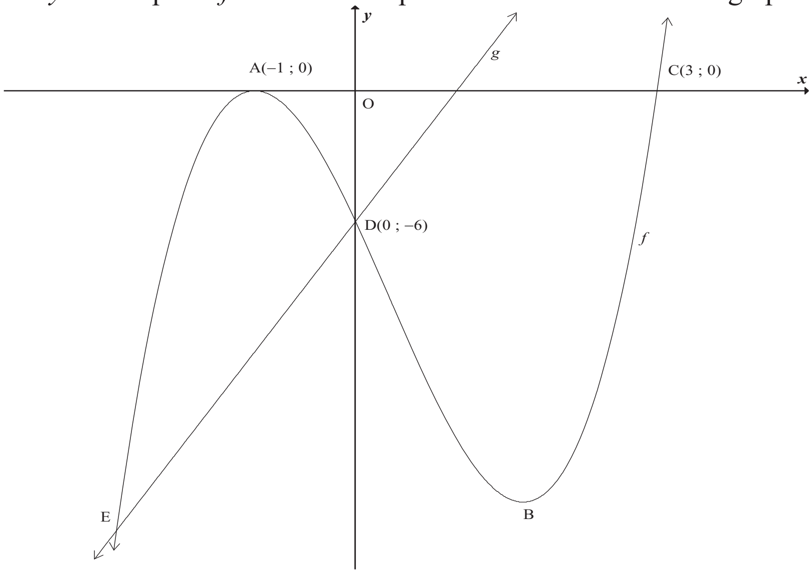


2.1.1 Calculate the x-coordinates of the turning points of f. (4)

2.1.2 Calculate the x-coordinate of the point at which $f'(x)$ is a maximum. (3)

2.1.3 Show that the concavity of f changes at $x = -\frac{1}{3}$ (3)

2.2 The graphs of $f(x) = ax^3 + bx^2 + cx + d$ and $g(x) = 6x - 6$ are sketched below. A(– 1 ; 0) and C(3 ; 0) are the x-intercepts of f. The graph of f has turning points at A and B. D(0 ; – 6) is the y-intercept of f. E and D are points of intersection of the graphs of f and g.



2.2.1 Show that $a = 2$; $b = -2$; $c = -10$ and $d = -6$. (5)

2.2.2 Calculate the coordinates of the turning point B. (5)

2.2.3 $h(x)$ is the vertical distance between $f(x)$ and $g(x)$, that is $h(x) = f(x) - g(x)$. (5)

Calculate x such that $h(x)$ is a maximum, where $x < 0$.. [15]

2.3 Consider the graph of $g(x) = -2x^2 - 9x + 5$.

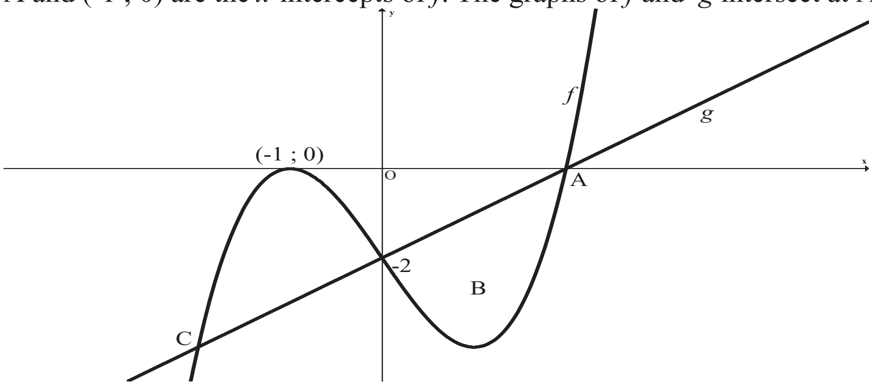
2.3.1 Determine the equation of the tangent to the graph of g at x = –1. (4)

2.3.2 For which values of q will the line y = –5x + q not intersect the parabola? (3)

2.4 Given: $h(x) = 4x^3 + 5x$

Explain if it is possible to draw a tangent to the graph of h that has a negative gradient. Show ALL your calculations. (3)

2.5 The graph below represents the function f and g with $f(x) = ax^3 - cx - 2$ and $g(x) = x - 2$. A and (-1 ; 0) are the x-intercepts of f. The graphs of f and g intersect at A and C.



2.5.1 Show by calculations that $a = 1$ and $c = -1$. (4)

2.5.2 Determine the coordinates of B, a turning point of f. (3)

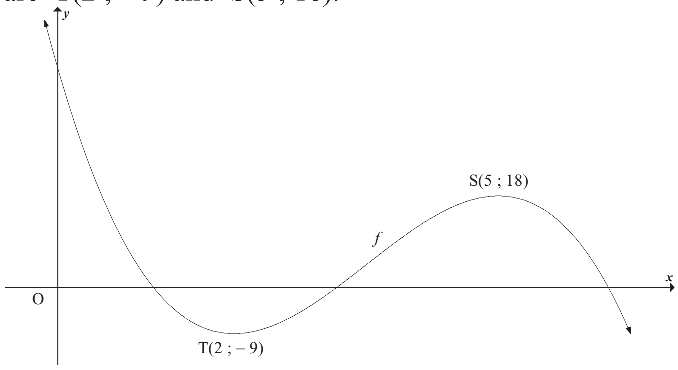
2.5.3 Show that the line BC is parallel to the x-axis. (7)

2.5.4 Find the x-coordinate of the point of inflection of f. (2)

2.5.5 Write down the values of k for which $f(x) = k$ will have only ONE root. (3)

2.5.6 Write down the values of x for which $x\, f'(x) < 0$. (2)

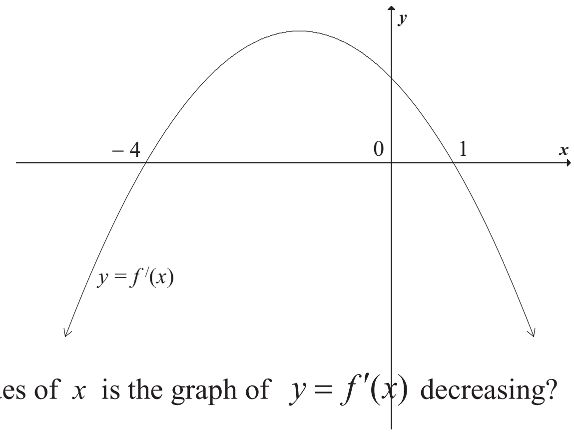
2.6 The function $f(x) = -2x^3 + ax^2 + bx + c$ is sketched below. The turning points of the graph of f are T(2 ; – 9) and S(5 ; 18).



2.6.1 Show that $a = 21$, $b = -60$ and $c = 43$. (7)

2.6.2 Determine an equation of the tangent to the graph of f at $x = 1$. (5)

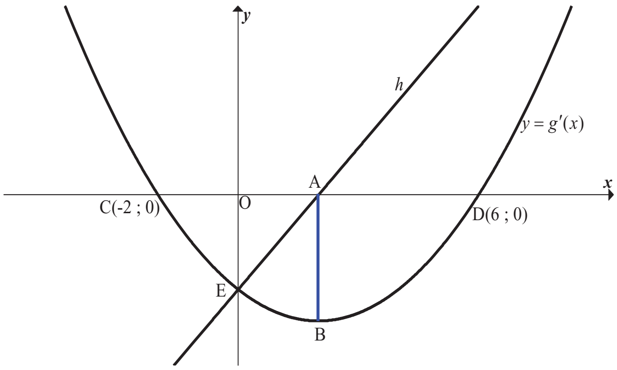
3.1 The graph of $y = f'(x)$, where f is a cubic function, is sketched below.



3.1.1 For which values of x is the graph of $y = f'(x)$ decreasing? (1)

3.1.2 At which value of x does the graph of f have a local minimum? Give reasons. (3)

3.2 The graphs of $y = g'(x) = ax^2 + bx + c$ and $h(x) = 2x - 4$ are sketched below. The graph of $y = g'(x) = ax^2 + bx + c$ is a derivative graph of a cubic function g. The graphs of h and g' have a common y-intercept at E. C(-2 ; 0) and D(6 ; 0) are the x-intercept of the graph of g'. A is the x-intercept of h and B is the turning point of g'. AB ‖ y-axis.



3.2.1 Write down the coordinates of E. (1)

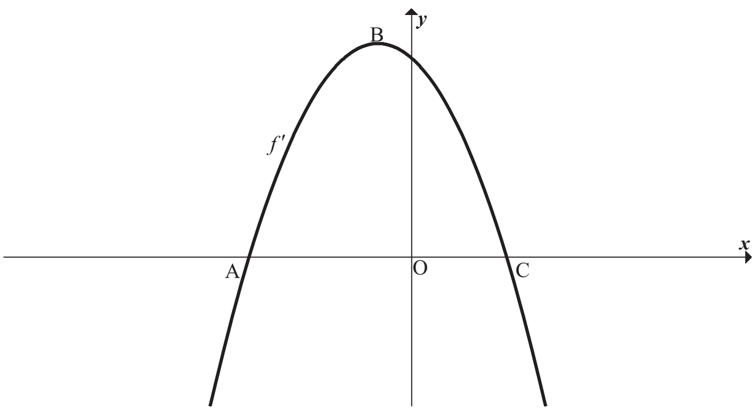
3.2.2 Determine the equation of the graph of g' in the form $y = ax^2 + bx + c$. (4)

3.2.3 Write down the x-coordinates of the turning points of g. (2)

3.2.4 Write down the x-coordinates of the point of inflection of the graph of g. (2)

3.2.5 Explain why g has a local maximum at $x = -2$ (3)

3.3 Sketched below is the graph of f', the derivative of $f(x) = -2x^3 - 3x^2 + 12x + 20$. A, B and C are the intercepts of f' with the axes.



3.3.1 Determine the coordinates of A. (2)

3.2.2 Determine the coordinates of B and C. (3)

3.2.3 Which points on the graph of $f(x)$ will have exactly the same x-coordinates as B and C? (1)

3.2.4 For which values of x will $f(x)$ be increasing? (2)

3.2.5 Determine the y-coordinates of the point of inflection of f. (4)